To review for the midterm

- Review your notes, the text, the homework problems, and the suggested exercises in the schedule.
- Do all true false questions in the text. The exam **will** include a significant T/F section.
- Below is a list of topics for the exam and a sampling of questions from past exams. Review these also.
- VERY IMPORTANT. These are just questions from old exams for your **practice**. The questions on our exam may not be similar.

Vector Spaces and Subspaces

- The axioms of vector spaces and fields.
- The vector space $\mathbb{F}^n$ associated to a field $\mathbb{F}$
- Proving and disproving that a set with two given operations is a vector space.
- The definition of a subspace.
- Proving or disproving that a subset of a vector space is a subspace.
- The sum, direct sum and intersection of subspaces.

Example Problems from old exams

(1) Let $V = \{(a, b) \mid a, b \in \mathbb{R}\}$ Define scalar multiplication $\boxdot$ and vector addition $\boxplus$ as follows:
For $\lambda \in \mathbb{R}$, $\lambda \boxdot (a, b) = (-\lambda b, \lambda a)$ and $(a_1, b_1) \boxplus (a_2, b_2) = (a_1 - a_2, b_1 + b_2)$ Prove that V is not a real vector space with these operations.

(2) Let V be a vector space over a field F, and let $T : V \to V$ be a linear transformation. Let $W = \{x \in V \mid T(x) = x\}$. Show that W is a subspace of V.

(3) Let V the space of all functions $f : \mathbb{R} \to \mathbb{R}$, and let W be the set of all functions f such that $f(1) = -f(2)$. Show that W is a subspace of V .

(4) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Let W be the set of all $v \in \mathbb{R}^2$ such that $T(v) = (1, 1, 1)$. Is W a subspace of $\mathbb{R}^2$? Explain your answer carefully.

(5) Let $V = M_{2\times 2}(\mathbb{R})$ and $W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$ and $W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$.
Determine a basis for and the dimension of the subspaces W_1, W_2, $W_1 \cap W_2$, and $W_1 + W_2$.

Spanning, independence, bases, and dimension

- The span of a set of vectors as a subspace, and determining if a given vector belongs to a span.
- Showing that a set of vectors is linearly independent or dependent. Finding linearly independent subsets.
- Showing that a set of vectors is a basis for a vector space.
- Identifying the dimensions of the most common vector spaces.
- The uniqueness of the representation of a vector as a linear combination of vectors in a chosen basis.
- The relation between linearly independent sets, spanning sets and bases.
- Finding a basis for a subspace.

(1) Suppose that $\{u, v\}$ is a basis for a vector space V . Show that $\{u + v, u + 2v\}$ is also a basis for V .
(2) Let V be a vector space and let $T : V \to V$ be a linear transformation.
(a) Suppose that $\{v_1, v_2\}$ is dependent. Show that $\{T(v_1), T(v_2)\}$ must also be dependent.
(b) True or false and explain: Suppose that $\{v_1, v_2\}$ is independent. Then $\{T(v_1), T(v_2)\}$ must also be independent.
(3) Let F be a field, and let W be the subspace of F^n defined as

$$W = \{(a_1, \dots, a_n) \mid a_1 + \cdots + a_n = 0\}.$$

Find the dimension of W, making sure to justify your work.
(4) Let V be a vector space of dimension $n \geq 2$. Let W_1 and W_2 be subspaces of V such that $W_1 \neq V, W_2 \neq V$, and $W_1 \neq W_2$. Show that $\dim(W_1 \cap W_2) \leq \dim(V) - 2$.
(5) Suppose $\{v_1, \dots, v_n\}$ is a linearly independent set in vector space V and $w \in V$. Prove that if $\{v_1 + w, \dots, v_n + w\}$ is linearly dependent, then $w \in \text{span}\{v_1, \dots, v_n\}$.

Linear Transformations

- Showing a transfromation is linear.
- Identifying nullspace and range.
- Using the dimension (rank-nullity) theorem.

(1) Suppose $T : V \to V$ is a linear transformation and that $\dim(V) = 3$. Give an example using a specific V and T such that $R(T) = N(T)$, or explain why this is not possible.
(2) Suppose $T : \mathbb{R}^5 \to P_5(\mathbb{R})$ is a linear map. Suppose there exists $x \in \mathbb{R}^5$ such that $x \neq \mathbf{0}$ and $T(x) = \mathbf{0}$. Can T be onto? If yes, give an example of such a T and corresponding non-zero x. If not, justify why not.
(3) Let $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^2$ be given by $T(A) = (a_{11} - a_{22}, -2a_{22} - a_{12})$.
(a) Show that T is linear.
(b) Find bases for the nullspace of T and for the range of T.
(4) (a) Give an example of vector spaces V and W and a linear map $T : V \to W$ such that T is one-to-one but not onto.
(b) Give an example of vector spaces V and W and a linear map $T : V \to W$ such that T is onto but not one-to-one. Page
(5) Let $V = P_2(\mathbb{R})$ and $W = P_3(\mathbb{R})$, and define $T : V \to W$ by $T(f(x)) = xf(x) + f'(x)$. Show that T is a linear transformation.